Critical Points

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review: a point $(x,y) = (a,b)$, a, b real values = critical for $f(x,y)$ if ...

all the partial derivatives vanish

$\partial_x f(a,b) = 0$

$\partial_y f(a,b) = 0$

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consider $f(x,y) = x^4 + y^4 - 4xy$

1) what are critical points?

hessian (2nd derivative)

2) for each critical point: min, max, saddle, not sure?

for 1), we compute $\partial_x f$ & $\partial_y f$ and solve 2 equations:

$\partial_x f = 0$ $\partial_y f = 0$ both / if only one vanishes, not critical

* linear in x & y: no critical

cos & sin: infinite critical *

- $\partial_x f = 4x^3 - 4y$
- $\partial_y f = 4y^3 - 4x$

$$\begin{cases} 4x^3 - 4y = 0 \\ 4y^3 - 4x = 0 \end{cases}$$

to solve: 1st implies $x^3 = y$, substitute in 2nd to get $x^9 - x = 0 \longrightarrow (x^8 - 1)x = 0$

- $x = 0$, then $y = x^3 = 0 \longrightarrow \boxed{(0,0)}$ = critical
- $x^8 - 1 = 0 \longrightarrow$ $x = 1 \longrightarrow y = 1^3 = 1 \longrightarrow \boxed{(1,1)}$ = critical
  $x = -1 \longrightarrow y = (-1)^3 = -1 \longrightarrow \boxed{(-1,-1)}$ = critical

$4x^3 - 4y = 0$
$x^3 - y = 0$
$x^3 = y$

$4y^3 - 4x = 0$
$y^3 - x = 0$
$(x^3)^3 - x = 0$
$x^9 - x = 0$

how to decide min, max, saddle?

- 2nd derivatives $\longrightarrow$ there are 4 of them

$\partial_x(\partial_y f)$ / derive partial derivative of y in respect to x

$\partial_{xx} f = 12x^2$

$\partial_{yy} f = 12y^2$

always equal

$\partial_{xy} f = -4$

$\partial_{yx} f = -4$

(x_0, y_0) = critical point / #'s

general recipe: suppose $a = \partial_{xx} f(x_0, y_0)$

$b = \partial_{xy} f(x_0, y_0)$

$c = \partial_{yx} f(x_0, y_0)$

$d = \partial_{yy} f(x_0, y_0)$

after substituting same point

* a & d need to be xx & yy *

* b & c need to be xy & yx *

ex) $(x_0, y_0) = (0,0)$

$(x_0, y_0) = (1,1)$

$(x_0, y_0) = (-1,-1)$

the characteristic polynomial @ (x_0, y_0) is polynomial:

$$\lambda^2 - (a+d)\lambda + (ad - bc) \longrightarrow$$ then solve for roots of λ by setting $= 0$

min if both roots > 0

saddle if one root > 0 & other < 0

decide: $\begin{cases} \text{min if both roots} > 0 \\ \text{saddle if one root} > 0 \text{ \& other} < 0 \\ \text{max if both roots} < 0 \end{cases}$

($= 0 \rightarrow$ no idea)

three critical:

(x_0, y_0)	a, b, c, d	char. polynomial	roots of polynomial	
$(0,0)$	$a = 12 \cdot 0^2 = 0$, $b = -4$, $c = -4$, $d = 12 \cdot 0^2 = 0$	$\lambda^2 - 0\lambda - 16$	4 & -4	saddle
$(1,1)$	$a = 12 \cdot 1^2 = 12$, $b = -4$, $c = -4$, $d = 12 \cdot 1^2 = 12$	$\lambda^2 - 24\lambda + 128$	8 & 16	min
$(-1,-1)$	$a = 12 \cdot -1^2 = 12$, $b = -4$, $c = -4$, $d = 12 \cdot -1^2 = 12$	$\lambda^2 - 24\lambda + 128$	8 & 16	min

$\lambda^2 - 16 = 0$

$(\lambda - 4)(\lambda + 4) = 0$

$\lambda = 4$ & -4

$\lambda^2 - 24\lambda + 128 = 0$

$(\lambda - 16)(\lambda - 8) = 0$

$\lambda = 16$ & 8

also could use:

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$